

$L(h, k)$ -labelings of Hamming graphs[☆]

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ABSTRACT

Given integers $c \geq 0$ and $h \geq k \geq 1$, a c - $L(h, k)$ -labeling of a graph G is a mapping $f: V(G) \rightarrow \{0, 1, 2, \dots, c\}$ such that $|f(u) - f(v)| \geq h$ if $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ if $d_G(u, v) = 2$. The $L(h, k)$ -number $\lambda_{h,k}(G)$ of G is the minimum c such that G has a c - $L(h, k)$ -labeling. The Hamming graph is the Cartesian product of complete graphs. In this paper, we study $L(h, k)$ -labeling numbers of Hamming graphs. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \leq q \leq p$ with $h/k \leq n - q + 1$ or $2 \leq q \leq p$ with $h/k \geq qn - 2q + 2$ or $q = p + 1$ with $h/k \leq n/p$, where p is the minimum prime factor of n .

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1. Introduction

The problem of labeling vertices with a condition at distance two was first studied by Griggs and Yeh [10]. It arose from a variation of the channel assignment problem introduced by Hale [11]. Given a number of transmitters (or stations), the object is to assign a channel to each transmitter, such that the interference is avoided. In order to reduce the interference, any two “close” transmitters must receive channels at least k apart, and any two “very close” transmitters must receive channels at least h apart, where $h \geq k$. One can construct an interference graph for this problem so that the transmitters are represented by the vertices of a graph and there is an edge between two “very close” transmitters, while two transmitters are “close” if the corresponding vertices are of distance two.

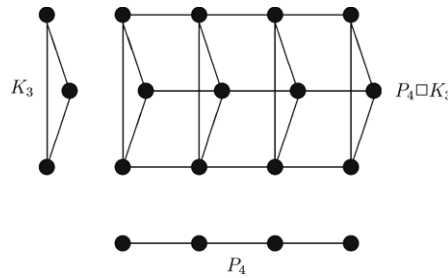
More precisely, for a given graph G and integers $h \geq k \geq 1$, an $L(h, k)$ -labeling of G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq h$ if $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ if $d_G(u, v) = 2$, where $d_G(u, v)$ is the distance between u and v in G . For a nonnegative integer c , a c - $L(h, k)$ -labeling is an $L(h, k)$ -labeling such that no label is greater than c . The $L(h, k)$ -labeling number of G , denoted by $\lambda_{h,k}(G)$, is the smallest c such that G has a c - $L(h, k)$ -labeling. A c - $L(h, k)$ -labeling f is optimal if $c = \lambda_{h,k}(G)$, and in this case f is also called a $\lambda_{h,k}$ -labeling.

The $L(h, k)$ -labeling problem has been studied in [3,4,6–10,12,14]. For more information, see the surveys [1,2,13]. The purpose of this paper is to study $L(h, k)$ -numbers of Hamming graphs described below. The Cartesian product of q graphs G_1, G_2, \dots, G_q is the graph $G_1 \square G_2 \square \dots \square G_q$ whose vertex set is $\prod_{i=1}^q V(G_i)$ and two vertices (u_1, u_2, \dots, u_q) and (v_1, v_2, \dots, v_q) are adjacent if and only if $u_j v_j \in E(G_j)$ for some j and $u_i = v_i$ for all other $i \neq j$. See Fig. 1 for an example of a Cartesian product of two graphs. For the case when $G_i = G$ for all i , we use G^q to denote $G_1 \square G_2 \square \dots \square G_q$. A Hamming graph is the Cartesian product $K_{n_1} \square K_{n_2} \square \dots \square K_{n_q}$ of complete graphs $K_{n_1}, K_{n_2}, \dots, K_{n_q}$.

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Fig. 1. $P_4 \square K_3$.

Georges, Mauro and Stein [9] used a group technique to determine the $\lambda_{2,1}$ -numbers of some special Hamming graphs: If $2 \leq q$ and p is a prime, then

$$\lambda_{2,1}(K_p^q) = p^{2r} - 1 \quad \text{if } \begin{cases} r \geq 2 \text{ and } q \leq p, & \text{(a)} \\ r = 1 \text{ and } q < p. & \text{(b)} \end{cases} \quad (1)$$

They also obtained that if p is prime then

$$\lambda_{1,1}(K_p^p) = \lambda_{1,1}(K_p^{p+1}) = p^2 - 1. \quad (2)$$

For the general $L(h, k)$ -labelings they proved that for integer $h \geq k \geq 1$,

$$\lambda_{h,k}(K_n \square K_m) = \begin{cases} (mn - 1)k, & \text{if } 2 \leq n < m \text{ and } h/k \leq n; & \text{(a)} \\ (m - 1)h + (n - 1)k, & \text{if } 2 \leq n < m \text{ and } h/k > n; & \text{(b)} \\ (n^2 - 1)k, & \text{if } 2 \leq n = m \text{ and } h/k \leq n - 1; & \text{(c)} \\ (n - 1)h + (2n - 2)k, & \text{if } 2 \leq n = m \text{ and } h/k > n - 1. & \text{(d)} \end{cases} \quad (3)$$

Besides some interesting results for $\lambda_{1,1}(G)$ on some Hamming graphs G , Georges and Mauro [8] also established that for integers $h \geq k \geq 1$,

$$\lambda_{h,k}(K_n^3) = \begin{cases} (n^2 - 1)k, & \text{if } n \text{ is odd and } h/k \leq n - 2; & \text{(a)} \\ \leq (n - 1)(h + 3k), & \text{if } n \text{ is odd and } h/k \geq n - 2; & \text{(b)} \\ = (n - 1)(h + 3k), & \text{if } n \text{ is odd and } h/k \geq 3n - 4; & \text{(c)} \\ (n^2 - 1)k, & \text{if } n \text{ is even and } h/k \leq n/2; & \text{(d)} \\ \leq (n - 1)h + n(2n - 1)k, & \text{if } n \text{ is even.} & \text{(e)} \end{cases} \quad (4)$$

Erwin et al. [6] determined the $\lambda_{h,k}$ -number of the Hamming graph $K_{n_1} \square K_{n_2} \square \cdots \square K_{n_q}$ for $q \geq 3$ and relatively prime n_1, n_2, \dots, n_q . Results for $\lambda_{h,k}$ -numbers of the Hamming graphs are also obtained when $(h, k) = (2, 0), (2, 1)$ and $(1, 1)$ in [5].

In this paper, we extend these results more generally. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \leq q \leq p$ with $h/k \leq n - q + 1$ or $2 \leq q \leq p$ with $h/k \geq qn - 2q + 2$ or $q = p + 1$ with $h/k \leq n/p$, where p is the minimum prime factor of n .

2. $\lambda_{h,k}$ -numbers of Hamming graphs K_n^q

In this section, we establish results for the $\lambda_{h,k}$ -numbers of Hamming graphs K_n^q , see Theorems 2 and 4. Suppose p is the minimum prime factor of n . Theorem 2 considers the case where $2 \leq q \leq p$, and Theorem 4 considers the case where $q = p + 1$ and $h/k \leq n/p$.

To fix notation, let $[n] = \{0, 1, \dots, n - 1\}$. We use $[n]$ as the vertex set of the complete graph K_n , and $[n]^q = \{(a_1, a_2, \dots, a_q) : a_i \in [n] \text{ for } 1 \leq i \leq q\}$ the vertex set of the Hamming graph K_n^q .

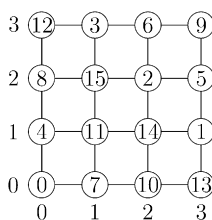
We first establish a lower bound for the result in Theorem 2 in the next lemma.

Lemma 1. If $n \geq 2$ and $q \geq 2$, then $\lambda_{h,k}(K_n^q) \geq (n - 1)(h + qk)$ whenever $h/k \geq qn - 2q + 2$.

Proof. Suppose to the contrary that K_n^q has an $L(h, k)$ -labeling f using labels only in $[0, (n - 1)(h + qk) - 1]$. Partition $[0, (n - 1)(h + qk) - 1]$ into segments $X_1, X_2, \dots, X_{(n-1)(q+1)}$ such that $|X_{iq+1}| = h$ for $0 \leq i \leq n - 2$ and all other $|X_r| = k$. Namely, with $w = h + (q - 1)k$ we have

$$X_{iq+j} = \begin{cases} [iw, iw + h - 1], & \text{if } 0 \leq i \leq n - 2 \text{ and } j = 1; \\ [iw + h + (j - 2)k, iw + h + (j - 1)k - 1], & \text{if } 0 \leq i \leq n - 2 \text{ and } 2 \leq j \leq q; \\ [(n - 1)w + (j - 1)k, (n - 1)w + jk - 1], & \text{if } i = n - 1 \text{ and } 1 \leq j \leq n - 1. \end{cases}$$

A segment X_r with $|X_r| = h$ is called a h -segment, and with $|X_r| = k$ a k -segment.

Fig. 2. The mapping f on K_4^2 .

We claim that no label under f is in a k -segment X_r with $r \leq (n-1)q$. Suppose to the contrary that there is a vertex v with $f(v) \in X_r$ for some k -segment X_r with $r \leq (n-1)q$. We may assume that r is such a minimum index. Write $r = sq + t$ where $0 \leq s \leq n-2$ and $2 \leq t \leq q$. Notice that K_n^q has q subgraphs isomorphic to K_n , each containing v . Each h -segment X_{iq+1} with $0 \leq i \leq s-1$ contains at most one label used by each copy of K_n and the h -segment X_{sq+1} contains one label from each of at most $t-1$ copies of K_n . Therefore, there are at least $q-t+1$ copies of K_n each contains at least $n-s-1$ vertices with label greater than $f(v)$. Thus, the largest label among those vertices is at least $f(v) + (n-s-1)h + (q-t)k \geq s(h + (q-1)k) + h + (t-2)k + (n-s-1)h + (q-t)k = nh + ((s+1)(q-1) - 1)k \geq nh + (q-2)k \geq (n-1)(h+qk)$, contradicting that all labels are in $[0, (n-1)(h+qk) - 1]$. Hence, no label under f is in a k -segment X_r with $r \leq (n-1)q$.

Now, $f^{-1}(X_r)$ contains at most n^{q-1} vertices for each h -segment X_r , and $f^{-1}(X_r)$ contains at most n^{q-2} vertices for each k -segment X_r with $r > (n-1)q$. Therefore, $|V(K_n^q)| \leq (n-1)n^{q-1} + (n-1)n^{q-2} = n^q - n^{q-2} < n^q$, a contradiction. \square

We now come to the first main result. Notice that (5)(a) generalizes (1)(a) and (b), first part of (2), (3)(c) and (4)(a); (5)(b) generalizes (4)(b); (5)(c) generalizes (4)(c). For instance, to see (5)(a) implying (1)(a), we let $h = 2$, $k = 1$ and $n = p^r$ with $r \geq 2$. In this case, condition " $h/k \leq n - q + 1$ " of (5)(a) and condition " $2 \leq q \leq p$ " of Theorem 2 give condition " $2 \leq q \leq p$ " of (1)(a). For (5)(a) implying (1)(b), we choose $h = 2$, $k = 1$ and $n = p^2$ with $r = 1$. In this case, condition " $h/k \leq n - q + 1$ " of (5)(a) is just condition " $q < p$ " of (1)(a). Similarly, we can check implications. We also notice that (5)(b) is only an upper bound. The exact value of $\lambda_{h,k}(K_n^q)$ for $n - q + 1 < h/k < qn - 2q + 2$ is desirable.

Theorem 2. If p is the minimum prime factor of $n \geq 2$ and $2 \leq q \leq p$, then

$$\lambda_{h,k}(K_n^q) \begin{cases} = (n^2 - 1)k, & \text{if } h/k \leq n - q + 1; & (a) \\ \leq (n-1)(h+qk), & \text{if } h/k > n - q + 1; & (b) \\ = (n-1)(h+qk), & \text{if } h/k \geq qn - 2q + 2. & (c) \end{cases} \quad (5)$$

Proof. For $a = (a_1, a_2, \dots, a_q) \in V(K_n^q)$, define

$$x(a) = \left(\sum_{i=1}^q a_i \right) \bmod n, \quad y(a) = \left(\sum_{i=1}^{q-1} -ia_i \right) \bmod n.$$

For the case of $h/k \leq n - q + 1$, consider the mapping f on $V(K_n^q)$ defined by

$$f(a) = (x(a)n + y(a))k.$$

See Fig. 2 for an example with $h = 3$, $k = 1$, $n = 4$, $q = 2$. We shall check that f is an $L(h, k)$ -labeling of K_n^q . Suppose $a = (a_1, a_2, \dots, a_q)$ and $b = (b_1, b_2, \dots, b_q)$ are two adjacent vertices in K_n^q , say they differ only at position r . It is then the case that $x(a) \neq x(b)$, say $x(a) > x(b)$. For the case of $x(a) \geq x(b) + 2$, we have $f(a) - f(b) \geq (n+1)k \geq h$. For the case of $x(a) = x(b) + 1$, we have $a_r = (b_r + 1) \bmod n$. If $r \leq q-1$, then $y(a) = (y(b) - r) \bmod n$ and so $f(a) - f(b) \geq (n-r)k \geq (n-q+1)k \geq h$. If $r = q$, then $y(a) = y(b)$, which gives that $f(a) - f(b) = nk \geq h$. We next consider the case when $a = (a_1, a_2, \dots, a_q)$ and $b = (b_1, b_2, \dots, b_q)$ are of distance two in K_n^q , say they differ only at positions r and s with $1 \leq r < s \leq q$. In order to check the distance two condition, we only have to verify that $x(a) \neq x(b)$ or $y(a) \neq y(b)$. Suppose to the contrary that $x(a) = x(b)$ and $y(a) = y(b)$. If $s \leq q-1$, then $a_r + a_s \equiv b_r + b_s \pmod{n}$ and $ra_r + sa_s \equiv rb_r + sb_s \pmod{n}$ lead to $(s-r)a_r \equiv (s-r)b_r \pmod{n}$. Since $1 \leq (s-r) < p$, so $a_r = b_r$, which is impossible. If $s = q$, then $ra_r \equiv rb_r \pmod{n}$. Since $1 \leq r < p$, so $a_r = b_r$, also impossible. Therefore, f is an $L(h, k)$ -labeling of K_n^q . This and the fact that the n^2 vertices whose r -th coordinate are 0 for $3 \leq r \leq q$ are pairwise of distance two gives that $(n^2 - 1)k \leq \lambda_{h,k}(K_n^q) \leq (n^2 - 1)k$ and so $\lambda_{h,k}(K_n^q) = (n^2 - 1)k$.

For the case of $h/k > n - q + 1$, consider the mapping g on $V(K_n^q)$ defined by

$$g(a) = x(a)(h + (q-1)k) + y(a)k.$$

We shall check that g is an $L(h, k)$ -labeling of K_n^q . Suppose $a = (a_1, a_2, \dots, a_q)$ and $b = (b_1, b_2, \dots, b_q)$ are two adjacent vertices in K_n^q , say they differ only at position r . It is then the case that $x(a) \neq x(b)$, say $x(a) > x(b)$. For the case of $x(a) \geq x(b) + 2$, we have $g(a) - g(b) \geq 2(h + (q-1)k) - (n-1)k = 2h - (n-1-2q+2)k \geq h$. For the case of $x(a) = x(b) + 1$, we have $a_r = (b_r + 1) \bmod n$. If $r \leq q-1$, then $y(a) = (y(b) - r) \bmod n$ and so $g(a) - g(b) \geq ((h + (q-1)k) - rk) \geq h$.

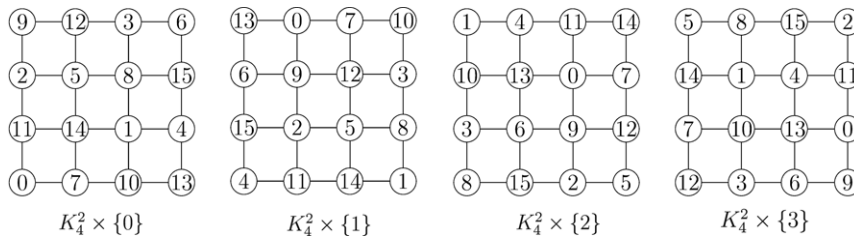


Fig. 3. A $\lambda_{2,1}$ -labeling of K_4^3 .

If $r = q$, then $y(a) = y(b)$, which give that $g(a) - g(b) = h + (q - 1)k \geq h$. Since $h + (q - 1)k \geq nk$, precisely the same argument as in the last case gives that $|g(a) - g(b)| \geq k$ for vertices a and b of distance two in K_n^q . Hence, $\lambda_{h,k}(K_n^q) \leq (n - 1)(h + qk)$ as desired.

The last equality follows from Lemma 1 and the inequality above. \square

For the second result, we need the following useful lemma.

Lemma 3. If p is the minimum prime factor of $n \geq 2$, $1 \leq r \leq p - 1$, $1 \leq s \leq n - 1$ and $rs \equiv 1 \pmod{n}$, then $s \leq n - n/p$.

Proof. Suppose to the contrary that $s = n - n/p + t$ for some $1 \leq t < n/p$. Then, $rs = rn - rn/p + rt \equiv 1 \pmod{n}$ and so $r(n/p - t) + 1 \equiv 0 \pmod{n}$, contradicting $1 \leq r(n/p - t) + 1 \leq (p - 1)(n/p - t) + 1 < n$. \square

Then, we have the second main result. Notice that it generalizes the second part of (2) and (4)(d).

Theorem 4. If p is the minimum prime factor of $n \geq 2$, then $\lambda_{h,k}(K_n^{p+1}) = (n^2 - 1)k$ whenever $h/k \leq n/p$.

Proof. For $a = (a_1, a_2, \dots, a_{p+1}) \in V(K_n^{p+1})$, define

$$x(a) = \left(a_{p+1} + \sum_{i=1}^p ia_i \right) \pmod{n}, \quad y(a) = \left(\sum_{i=1}^p -a_i \right) \pmod{n}.$$

We consider the mapping f on $V(K_n^{p+1})$ defined by

$$f(a) = (x(a)n + y(a))k.$$

See Fig. 3 for an example with $h = 2$, $k = 1$, $n = 4$, $p = 2$. We shall check that f is an $L(h, k)$ -labeling of K_n^{p+1} . Suppose $a = (a_1, a_2, \dots, a_{p+1})$ and $b = (b_1, b_2, \dots, b_{p+1})$ are two adjacent vertices in K_n^{p+1} , say they differ only at position r and let $\Delta_r = (a_r - b_r) \pmod{n}$. We may assume that $x(a) \geq x(b)$. For the case of $x(a) \geq x(b) + 2$, we have $f(a) - f(b) \geq (n + 1)k \geq h$. For the case of $x(a) = x(b) + 1$, we have $r \neq p$ and $s\Delta_r \equiv 1 \pmod{n}$ for some $1 \leq s \leq p - 1$. By Lemma 3, $f(a) - f(b) \geq (n - n + n/p)k \geq nk/p \geq h$. For the case of $x(a) = x(b)$, we have $r = p$ and $\Delta_r = sn/p \pmod{n}$ for some $1 \leq s < n/p$. Therefore, $f(a) - f(b) \geq nk/p \geq h$.

We next consider the case when $a = (a_1, a_2, \dots, a_{p+1})$ and $b = (b_1, b_2, \dots, b_{p+1})$ are of distance two in K_n^{p+1} , say they differ only at positions r and s with $1 \leq r < s \leq p + 1$. In order to check the distance two condition, we only have to verify that $x(a) \neq x(b)$ or $y(a) \neq y(b)$. Suppose to the contrary that $x(a) = x(b)$ and $y(a) = y(b)$. If $s \leq p$, then $a_r + a_s \equiv b_r + b_s \pmod{n}$ and $ra_r + sa_s \equiv rb_r + sb_s \pmod{n}$ lead to $(s - r)a_r \equiv (s - r)b_r \pmod{n}$. Since $1 \leq (s - r) < p$, so $a_r = b_r$, which is impossible. If $s = p + 1$, then $a_r = b_r$, also impossible.

Therefore, f is an $L(h, k)$ -labeling of K_n^{p+1} . This and the fact that the n^2 vertices whose r -th coordinate are 0 for $3 \leq r \leq p + 1$ are pairwise of distance two gives that $(n^2 - 1)k \leq \lambda_{h,k}(K_n^{p+1}) \leq (n^2 - 1)k$ and so $\lambda_{h,k}(K_n^{p+1}) = (n^2 - 1)k$. \square

3. Conclusion remarks and open problem

The purpose of this paper is to study $L(h, k)$ -labeling numbers of Hamming graphs. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \leq q \leq p$ with $h/k \leq n - q + 1$ or $2 \leq q \leq p$ with $h/k \geq qn - 2q + 2$ or $q = p + 1$ with $h/k \leq n/p$, where p is the minimum prime factor of n . Although these generalize previous results to some degree, results for more general cases are still desirable and should be studied further.

For instance, in the case when $2 \leq q \leq p$ and $n - q + 1 < h/k < qn - 2q + 2$, we only have the upper bound $\lambda_{h,k}(K_n^q) \leq (n - 1)(h + qk)$. Although we believe that this is the exact value, an argument stronger than that in the proof of Lemma 1 is needed to establish the lower bound.

Having Theorem 4, we are still lacking the value of $\lambda_{h,k}(K_n^q)$ for the case when $q = p + 1$ and $h/k > n/p$. More generally, results for the case of $q > p + 1$ are desirable, although it is not expected to be easy.

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